2025

2nd Semester Examination (CCFUP : NEP)

MATHEMATICS

Paper : MJ 2-T (Single Core Major)

[Algebra]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

Answer any *ten* questions : 2×10=20

1. Find the values of $(1+i)^{\frac{1}{5}}$.

2. Prove that if $2^n - 1$ is prime, then n is prime.

3. If $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and if p is prime to n, prove that $1+\alpha^p+\alpha^{2p}+\ldots..+\alpha^{(n-1)p}=0$.

4. If a, b, c are all positive real numbers and $a+b+c=1$, prove that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} \geq \frac{9}{2}$.

P.T.O.

5. Let R_1 and R_2 be equivalence relations on a set S ... that $R_1 \circ R_2 = R_2 \circ R_1$. Prove that $R_1 \circ R_2$ i... equivalence relation.

6. Is $f^{-1}(f(A)) = A$ always true? Justify.

7. Show that the rank of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \beta \\ 0 & 0 & 0 \end{pmatrix}$; $\alpha, \beta \in \mathbb{R}$

is independent of the values of α and β.

8. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Then find the eigen values of $A^3 + 4A = 5I$.

9. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$, where $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

10. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $Range(T) = Null(T)$.

11. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$. Find the conditions on b and c under which T is linear.

12. Find the last two digits in 7^{100}.

13. Verify whether two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent or not.

14. If A and B are two square matrices of order n, prove that $trace(AB) = trace(BA)$.

...r the equivalence relation ρ on $\mathbb{Z}$, defined by "$a\rho b$ if and only if $5 \mid a - b$ for $a, b \in \mathbb{Z}$", deduce all equivalence classes.

Group - B

Answer any ***four*** questions : $5 \times 4 = 20$

16. If each a, b, c, d be greater than 1 then show that $8(abcd + 1) > (a+1)(b+1)(c+1)(d+1)$.

17. By the principle of mathematical induction, prove that $3^{2n+1} + (-1)^n 2 \equiv 0 (\bmod 5)$, $\forall n \in \mathbb{N}$.

18. If z be a complex number and $\frac{z+1}{z-i}$ be purely imaginary, then show that z lies on the circle whose centre is at $\frac{1}{2}(-1+i)$ and the radius is $\frac{1}{\sqrt{2}}$.

19. If one of the roots of the equation $x^3 + px^2 + qx + r = 0$ equals the sum of the other two, then prove that $p^3 + 8r = 4pq$.

20. Find the values of k for which the system of equations

$$x + y - z = 1$$
$$2x + 3y + kz = 3$$
$$x + ky + 3z = 2$$

has (i) no solution (ii) more than one solutions (iii) unique solution.

P.T.O.

21. Define $\gcd(a,b)$ where a, b are integers. Prove that $\gcd(a^2,b^2)=(\gcd(a,b))^2$ for positive integers a,b.

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Group - C

Answer any *two* questions : 10×2=20

22. (i) Reduce the matrix $A=\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row-reduced Echelon form and hence find its rank.

(ii) For a linear transformation $T:P_2(\mathbb{R})\to P_2(\mathbb{R})$ defined by $T(p(x))=p(x)+(x+1)p'(x)$, compute matrix representation of T and hence find eigenvalues of T.

5+5

23. (i) Find a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ such that $Im(T)$ is the subspace

$$U=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}.$$

(ii) Use Cayley-Hamilton theorem, to find A^{50} where

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}.$$

(iii) Find $k\in\mathbb{R}$, under which the set

$$S=\{(k,1,1,1),(1,k,1,1),(1,1,k,1),(1,1,1,k)\}$$

is linearly independent in the Vector space $\mathbb{R}^4$.

4+4+2

(i) The matrix of a linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ relative to the ordered basis

$\{(-1,1,1),(1,-1,1),(1,1,-1)\}$ of $\mathbb{R}^3$ is $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$.

Find T.

(ii) If $2\cos\theta=t$, prove that

$$\frac{1+\cos 7\theta}{1+\cos\theta}=\left(t^3-t^2-2t+1\right)^2.$$

(iii) Find the minimum value of $x^2+y^2+z^2$ where x,y,z are positive real variables satisfying

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

4+3+3

25. (i) Show that if p and q are distinct primes, then $p^{q-1}+q^{p-1}\equiv 1 \pmod{pq}$.

(ii) Solve by Cardan's method : $x^3-3x+1=0$.

(iii) Without determining the det(A), conclude whether A is invertible or not.

3+4+3